

Structure of optimal strategies for remote estimation over Gilbert-Elliott channel with feedback

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Abstract—We investigate remote estimation over a Gilbert-Elliott channel with feedback. We assume that the channel state is observed by the receiver and fed back to the transmitter with one unit delay. In addition, the transmitter gets ACK/NACK feedback for successful/unsuccessful transmission. Using ideas from team theory, we establish the structure of optimal transmission and estimation strategies and identify a dynamic program to determine optimal strategies with that structure. We then consider first-order autoregressive sources where the noise process has unimodal and symmetric distribution. Using ideas from majorization theory, we show that the optimal transmission strategy has a threshold structure and the optimal estimation strategy is Kalman-like.

I. INTRODUCTION

A. Motivation and literature overview

We consider a remote estimation system in which a sensor/transmitter observes a first-order Markov process and causally decides which observations to transmit to a remotely located receiver/estimator. Communication is expensive and takes place over a Gilbert-Elliott channel (which is used to model channels with burst erasures). The channel has two states: OFF state and ON state. When the channel is in the OFF state, a packet transmitted from the sensor to the receiver is dropped. When the channel is in the ON state, a packet transmitted from the sensor to the receiver is received without error. We assume that the channel state is causally observed at the receiver and is fed back to the transmitter with one-unit delay. Whenever there is a successful reception, the receiver sends an acknowledgment to the transmitter. The feedback is assumed to be noiseless.

At the time instances when the receiver does not receive a packet (either because the sensor did not transmit or because the transmitted packet was dropped), the receiver needs to estimate the state of the source process. There is a fundamental trade-off between communication cost and estimation accuracy. Transmitting all the time minimizes the estimation error but incurs a high communication cost; not transmitting at all minimizes the communication cost but incurs a high estimation error.

The motivation of remote estimation comes from networked control systems. The earliest instance of the problem was perhaps considered by Marschak [1] in the context of information gathering in organizations. In recent years, several variations of remote estimation has been considered. These include models

that consider idealized channels without packet drops [2]–[9] and models that consider channels with i.i.d. packet drops [10], [11].

The salient features of remote estimation are as follows:

- (F1) The decisions are made sequentially.
- (F2) The reconstruction/estimation at the receiver must be done with zero-delay.
- (F3) When a packet does get through, it is received without noise.

Remote estimation problems may be viewed as a special case of real-time communication [12]–[15]. As in real-time communication, the key conceptual difficulty is that the data available at the transmitter and the receiver is increasing with time. Thus, the domain of the transmission and the estimation function increases with time.

To circumvent this difficulty one needs to identify sufficient statistics for the data at the transmitter and the data at the receiver. In the real-time communication literature, dynamic team theory (or decentralized stochastic control theory) is used to identify such sufficient statistics as well as to identify a dynamic program to determine the optimal transmission and estimation strategies. Similar ideas are also used in remote-estimation literature. In addition, feature (F3) allows one to further simplify the structure of optimal transmission and estimation strategies. In particular, when the source is a first-order autoregressive process, majorization theory is used to show that the optimal transmission strategies is characterized by a threshold [5]–[7], [10], [11]. In particular, it is optimal to transmit when the instantaneous distortion due to not transmitting is greater than a threshold. The optimal thresholds can be computed either using dynamic programming [5], [6] or using renewal relationships [10], [16].

All of the existing literature on remote-estimation considers either channels with no packet drops or channels with i.i.d. packet drops. In this paper, we consider packet drop channels with Markovian memory. We identify sufficient statistics at the transmitter and the receiver. When the source is a first-order autoregressive process, we show that threshold-based strategies are optimal but the threshold depends on the previous state of the channel.

B. The communication system

1) *Source model*: The source is a first-order time-homogeneous Markov process $\{X_t\}_{t \geq 0}$, $X_t \in \mathcal{X}$. For ease of exposition, in the first part of the paper we assume that \mathcal{X} is a finite set. We will later argue that a similar argument works when \mathcal{X} is a general measurable space. The transition probability matrix of the source is denoted by P , i.e., for any $x, y \in \mathcal{X}$,

$$P_{xy} := \mathbb{P}(X_{t+1} = y \mid X_t = x).$$

2) *Channel model*: The channel is a Gilbert-Elliott channel [17], [18]. The channel state $\{S_t\}_{t \geq 0}$ is a binary-valued first-order time-homogeneous Markov process. We use the convention that $S_t = 0$ denotes that the channel is in the OFF state and $S_t = 1$ denotes that the channel is in the ON state. The transition probability matrix of the channel state is denoted by Q , i.e., for $r, s \in \{0, 1\}$,

$$Q_{rs} := \mathbb{P}(S_{t+1} = s \mid S_t = r).$$

The input alphabet $\bar{\mathcal{X}}$ of the channel is $\mathcal{X} \cup \{\mathfrak{E}\}$, where \mathfrak{E} denotes the event that there is no transmission. The channel output alphabet \mathcal{Y} is $\mathcal{X} \cup \{\mathfrak{E}_0, \mathfrak{E}_1\}$, where the symbols \mathfrak{E}_0 and \mathfrak{E}_1 are explained below. At time t , the channel input is denoted by \bar{X}_t and the channel output is denoted by Y_t .

The channel is a channel with state. In particular, for any realization $(\bar{x}_{0:T}, s_{0:T}, y_{0:T})$ of $(\bar{X}_{0:T}, S_{0:T}, Y_{0:T})$, we have that

$$\begin{aligned} \mathbb{P}(Y_t = y_t \mid \bar{X}_{0:t} = \bar{x}_{0:t}, S_{0:t} = s_{0:t}) \\ = \mathbb{P}(Y_t = y_t \mid \bar{X}_t = \bar{x}_t, S_t = s_t) \end{aligned} \quad (1)$$

and

$$\begin{aligned} \mathbb{P}(S_t = s_t \mid \bar{X}_{0:t} = \bar{x}_{0:t}, S_{0:t-1} = s_{0:t-1}) \\ = \mathbb{P}(S_t = s_t \mid S_{t-1} = s_{t-1}) = Q_{s_{t-1}s_t} \end{aligned} \quad (2)$$

Note that the channel output Y_t is a deterministic function of the input \bar{X}_t and the state S_t . In particular, for any $\bar{x} \in \bar{\mathcal{X}}$ and $s \in \{0, 1\}$, the channel output y is given as follows:

$$y = \begin{cases} \bar{x}, & \text{if } \bar{x} \in \mathcal{X} \text{ and } s = 1 \\ \mathfrak{E}_1, & \text{if } \bar{x} = \mathfrak{E} \text{ and } s = 1 \\ \mathfrak{E}_0, & \text{if } s = 0 \end{cases}$$

This means that if there is a transmission (i.e., $\bar{x} \in \mathcal{X}$) and the channel is on (i.e., $s = 1$), then the receiver observes \bar{x} . However, if there is no transmission (i.e., $\bar{x} = \mathfrak{E}$) and the channel is on (i.e., $s = 1$), then the receiver observes \mathfrak{E}_1 , if the channel is off, then the receiver observes \mathfrak{E}_0 .

3) *The transmitter*: There is no need for channel coding in a remote-estimation setup. Instead, the role of the transmitter is to determine which source realizations need to be transmitted. Let $U_t \in \{0, 1\}$ denote the transmitter's decision. We use the convention that $U_t = 0$ denotes that there is no transmission (i.e., $\bar{X}_t = \mathfrak{E}$) and $U_t = 1$ denotes that there is transmission (i.e., $\bar{X}_t = X_t$).

Transmission is costly. Each time the transmitter transmits (i.e., $U_t = 1$), it incurs a cost of λ .

4) *The receiver*: At time t , the receiver generates an estimate $\hat{X}_t \in \mathcal{X}$ of X_t . The quality of the estimate is determined by a distortion function $d: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$.

C. Information structure and problem formulation

It is assumed that the receiver observes the channel state causally. Thus, the information available at the receiver¹ is

$$I_t^2 = \{S_{0:t}, Y_{0:t}\}.$$

The estimate \hat{X}_t is chosen according to

$$\hat{X}_t = g_t(I_t^2) = g_t(S_{0:t}, Y_{0:t}), \quad (3)$$

where g_t is called the *estimation rule* at time t . The collection $\mathbf{g} := (g_1, \dots, g_T)$ for all time is called the *estimation strategy*.

It is assumed that there is one-step delayed feedback from the receiver to the transmitter.² Thus, the information available at the transmitter is

$$I_t^1 = \{X_{0:t}, U_{0:t-1}, S_{0:t-1}, Y_{0:t-1}\}.$$

The transmission decision U_t is chosen according to

$$U_t = f_t(I_t^1) = f_t(X_{0:t}, U_{0:t-1}, S_{0:t-1}, Y_{0:t-1}), \quad (4)$$

where f_t is called the *transmission rule* at time t . The collection $\mathbf{f} := (f_1, \dots, f_T)$ for all time is called the *transmission strategy*.

The collection (\mathbf{f}, \mathbf{g}) is called a *communication strategy*. The performance of any communication strategy (\mathbf{f}, \mathbf{g}) is given by

$$J(\mathbf{f}, \mathbf{g}) = \mathbb{E} \left[\sum_{t=0}^T \lambda U_t + d(X_t, \hat{X}_t) \right] \quad (5)$$

where the expectation is taken with respect to the joint measure on all system variables induced by the choice of (\mathbf{f}, \mathbf{g}) .

We are interested in the following optimization problem.

Problem 1 *In the model described above, identify a communication strategy $(\mathbf{f}^*, \mathbf{g}^*)$ that minimizes the cost $J(\mathbf{f}, \mathbf{g})$ defined in (5).*

II. MAIN RESULTS

A. Structure of optimal communication strategies

Two-types of structural results are established in the real-time communication literature: (i) establishing that part of the data at the transmitter is irrelevant and can be dropped without any loss of optimality; (ii) establishing that the common information between the transmitter and the receiver can be “compressed” using a belief state. The first structural results were first established by Witsenhausen [12] while the second structural results were first established by Walrand Varaiya [13].

¹We use superscript 1 to denote variables at the transmitter and superscript 2 to denote variables at the receiver.

²Note that feedback requires two bits: the channel state S_t is binary and the channel output Y_t can be communicated by indicating whether $Y_t \in \mathcal{X}$ or not (i.e., transmitting an ACK or a NACK).

We establish both types of structural results for remote estimation. First, we show that $(X_{0:t-1}, U_{0:t-1})$ is irrelevant at the transmitter (Lemma 1); then, we use the common information approach of [19] and establish a belief-state for the common information $(S_{0:t}, Y_{0:t})$ between the transmitter and the receiver (Theorem 1).

Lemma 1 *For any estimation strategy of the form (3), there is no loss of optimality in restricting attention to transmission strategies of the form*

$$U_t = f_t(X_t, S_{0:t-1}, Y_{0:t-1}). \quad (6)$$

The proof idea is similar to [14]. We show that $\{X_t, S_{0:t-1}, Y_{0:t-1}\}_{t \geq 0}$ is a controlled Markov process controlled by $\{U_t\}_{t \geq 0}$. See Section III for proof.

Now, following [19], for any transmission strategy \mathbf{f} of the form (6) and any realization $(s_{0:T}, y_{0:T})$ of $(S_{0:T}, Y_{0:T})$, define $\varphi_t: \mathcal{X} \rightarrow \{0, 1\}$ as

$$\varphi_t(x) = f_t(x, s_{0:t-1}, y_{0:t-1}), \quad \forall x \in \mathcal{X}.$$

Furthermore, define conditional probability measures π_t^1 and π_t^2 on \mathcal{X} as follows: for any $x \in \mathcal{X}$,

$$\begin{aligned} \pi_t^1(x) &:= \mathbb{P}^{\mathbf{f}}(X_t = x \mid S_{0:t-1} = s_{0:t-1}, Y_{0:t-1} = y_{0:t-1}), \\ \pi_t^2(x) &:= \mathbb{P}^{\mathbf{f}}(X_t = x \mid S_{0:t} = s_{0:t}, Y_{0:t} = y_{0:t}). \end{aligned}$$

We call π_t^1 the *pre-transmission belief* and π_t^2 the *post-transmission belief*. Note that when $(S_{0:T}, Y_{0:T})$ are random variables, then π_t^1 and π_t^2 are also random variables which we denote by Π_t^1 and Π_t^2 .

For the ease of notation, for any $\varphi: \mathcal{X} \rightarrow \{0, 1\}$ and $i \in \{0, 1\}$, define the following:

- $B_i(\varphi) = \{x \in \mathcal{X} : \varphi(x) = i\}$.
- For any probability distribution π on \mathcal{X} and any subset \mathcal{A} of \mathcal{X} , $\pi(\mathcal{A})$ denotes $\sum_{x \in \mathcal{A}} \pi(x)$.
- For any probability distribution π on \mathcal{X} , $\xi = \pi|_{\varphi}$ means that $\xi(x) = \mathbb{1}_{\{\varphi(x)=0\}} \pi(x) / \pi(B_0(\varphi))$.

Lemma 2 *Given any transmission strategy \mathbf{f} of the form (6):*

- 1) *there exists a function F^1 such that*

$$\pi_{t+1}^1 = F^1(\pi_t^2) = \pi_t^2 P. \quad (7)$$

- 2) *there exists a function F^2 such that*

$$\pi_t^2 = F^2(\pi_t^1, \varphi_t, y_t). \quad (8)$$

In particular,

$$\pi_t^2 = \begin{cases} \delta_{y_t} & \text{if } y_t \in \mathcal{X} \\ \pi_t^1|_{\varphi_t}, & \text{if } y_t = \mathfrak{C}_1 \\ \pi_t^1, & \text{if } y_t = \mathfrak{C}_0. \end{cases} \quad (9)$$

Note that in (7), we are treating π_t^2 as a row-vector and in (9), δ_{y_t} denotes a Dirac measure centered at y_t . The update equations (7) and (8) are standard non-linear filtering equations. See Section III for proof.

Theorem 1 *In Problem 1, we have that:*

- 1) *Structure of optimal strategies: There is no loss of optimality in restricting attention to optimal transmission and estimation strategies of the form:*

$$U_t = f_t^*(X_t, S_{t-1}, \Pi_t^1), \quad (10)$$

$$\hat{X}_t = g_t^*(\Pi_t^2). \quad (11)$$

- 2) *Dynamic program: Let $\Delta(\mathcal{X})$ denote the space of probability distributions on \mathcal{X} . Define value functions $V_t^1: \{0, 1\} \times \Delta(\mathcal{X}) \rightarrow \mathbb{R}$ and $V_t^2: \{0, 1\} \times \Delta(\mathcal{X}) \rightarrow \mathbb{R}$ as follows.*

$$V_{T+1}^1(s, \pi^1) = 0, \quad (12)$$

and for $t \in \{T, \dots, 0\}$

$$\begin{aligned} V_t^1(s, \pi^1) &= \min_{\varphi: \mathcal{X} \rightarrow \{0, 1\}} \left\{ \lambda \pi^1(B_1(\varphi)) \right. \\ &\quad \left. + W_t^0(\pi^1, \varphi) \pi^1(B_0(\varphi)) + \sum_{x \in B_1(\varphi)} W_t^1(\pi^1, \varphi, x) \pi^1(x) \right\} \end{aligned} \quad (13)$$

$$V_t^2(s, \pi^2) = \min_{\hat{x} \in \mathcal{X}} \sum_{x \in \mathcal{X}} d(x, \hat{x}) \pi^2(x) + V_{t+1}^1(s, \pi^2 P), \quad (14)$$

where,

$$\begin{aligned} W_t^0(\pi^1, \varphi) &= Q_{s0} V_t^2(0, \pi^1) + Q_{s1} V_t^2(1, \pi^1|_{\varphi}), \\ W_t^1(\pi^1, \varphi, x) &= Q_{s0} V_t^2(0, \pi^1) + Q_{s1} V_t^2(1, \delta_x). \end{aligned}$$

Let $\Psi_t(s, \pi^1)$ denote the arg min of the right hand side of (13). Then, the optimal transmission strategy of the form (10) is given by

$$f_t^*(\cdot, s, \pi^1) = \Psi_t(s, \pi^1).$$

Furthermore, the optimal estimation strategy of the form (11) is given by

$$g_t^*(\pi^2) = \arg \min_{\hat{x} \in \mathcal{X}} \sum_{x \in \mathcal{X}} d(x, \hat{x}) \pi^2(x). \quad (15)$$

The proof idea is as follows. Once we restrict attention to transmission strategies of the form (6), the information structure is partial history sharing [19]. Thus, one can use the common information approach of [19] and obtain the structure of optimal strategies. See Section III for proof.

Remark 1 The first term in (13) is the expected communication cost, the second term is the expected cost-to-go when the transmitter does not transmit, and the third term is the expected cost-to-go when the transmitter transmits. The first term in (14) is the expected distortion and the second term is the expected cost-to-go.

Remark 2 Although the above model and result are stated for sources with finite alphabets, they extend naturally to general state spaces (including Euclidean spaces) under standard technical assumptions. See [20] for details.

B. Optimality of threshold-based strategies for autoregressive source

In this section, we consider a first-order autoregressive source $\{X_t\}_{t \geq 0}$, $X_t \in \mathbb{R}$, where the initial state $X_0 = 0$ and for $t \geq 0$, we have that

$$X_{t+1} = aX_t + W_t, \quad (16)$$

where $a \in \mathbb{R}$ and $W_t \in \mathbb{R}$ is distributed according to a symmetric and unimodal distribution with probability density function μ . Furthermore, the per-step distortion is given by $d(X_t - \hat{X}_t)$, where $d(\cdot)$ is an even function that is increasing on $\mathbb{R}_{\geq 0}$. The rest of the model is the same as before.

For the above model, we can further simplify the result of Theorem 1. See Section IV for the proof.

Theorem 2 *For a first-order autoregressive source with symmetric and unimodal disturbance,*

- 1) Structure of optimal estimation strategy: *The optimal estimation strategy is given as follows: $\hat{X}_0 = 0$, and for $t \geq 0$,*

$$\hat{X}_t = \begin{cases} a\hat{X}_{t-1}, & \text{if } Y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\} \\ Y_t, & \text{if } Y_t \in \mathbb{R} \end{cases} \quad (17)$$

- 2) Structure of optimal transmission strategy: *There exist threshold functions $k_t: \{0, 1\} \rightarrow \mathbb{R}_{\geq 0}$ such that the following transmission strategy is optimal:*

$$f_t(X_t, S_{t-1}, \Pi_t^1) = \begin{cases} 1, & \text{if } |X_t - a\hat{X}_{t-1}| \geq k_t(S_{t-1}) \\ 0, & \text{otherwise.} \end{cases} \quad (18)$$

Remark 3 As long as the receiver can distinguish between the events \mathfrak{E}_0 (i.e., $S_t = 0$) and \mathfrak{E}_1 (i.e., $U_t = 0$ and $S_t = 1$), the structure of the optimal estimator does not depend on the channel state information at the receiver.

Remark 4 It can be shown that under the optimal strategy, Π_t^2 is symmetric and unimodal around \hat{X}_t and, therefore, Π_t^1 is symmetric and unimodal around $a\hat{X}_{t-1}$. Thus, the transmission and estimation strategies in Theorem 2 depend on the pre- and post-transmission beliefs only through their means.

Remark 5 Recall that the distortion function is even and increasing. Therefore, the condition $|X_t - a\hat{X}_{t-1}| \geq k_t(S_{t-1})$ can be written as $d(X_t - a\hat{X}_{t-1}) \geq d(k_t(S_{t-1})) := d(k_t(S_{t-1}))$. Thus, the optimal strategy is to transmit if the per-step distortion due to not transmitting is greater than a threshold.

III. PROOF OF THE STRUCTURAL RESULTS

A. Proof of Lemma 1

Arbitrarily fix the estimation strategy g and consider the *best response* strategy at the transmitter. We will show that $\tilde{I}_t^1 := (X_t, S_{0:t-1}, Y_{0:t-1})$ is an information state at the transmitter.

Given any realization $(x_{0:T}, s_{0:T}, y_{0:T}, u_{0:T})$ of the system variables $(X_{0:T}, S_{0:T}, Y_{0:T}, U_{0:T})$, define $i_t^1 = (x_{0:t}, s_{0:t-1}, y_{0:t-1}, u_{0:t-1})$ and $\tilde{i}_t^1 = (x_t, s_{0:t-1}, y_{0:t-1})$.

Now, for any $\tilde{i}_{t+1}^1 = (\tilde{x}_{t+1}, \tilde{s}_{0:t}, \tilde{y}_{0:t}) = (\tilde{x}_{t+1}, \tilde{s}_t, \tilde{y}_t, \tilde{i}_t^1)$, we use the shorthand $\mathbb{P}(\tilde{i}_{t+1}^1 | \tilde{i}_{0:t}^1, u_{0:t})$ to denote $\mathbb{P}(\tilde{I}_{t+1}^1 = \tilde{i}_{t+1}^1 | \tilde{I}_{0:t}^1 = \tilde{i}_{0:t}^1, U_{0:t} = u_{0:t})$. Then,

$$\begin{aligned} \mathbb{P}(\tilde{i}_{t+1}^1 | i_t^1, u_t) &= \mathbb{P}(\tilde{x}_{t+1}, \tilde{s}_t, \tilde{y}_t, \tilde{i}_t^1 | x_{0:t}, s_{0:t-1}, y_{0:t-1}, u_{0:t}) \\ &\stackrel{(a)}{=} \mathbb{P}(\tilde{x}_{t+1}, \tilde{s}_t, \tilde{y}_t, \tilde{i}_t^1 | x_{0:t}, \bar{x}_{0:t}, s_{0:t-1}, y_{0:t-1}, u_{0:t}) \\ &\stackrel{(b)}{=} \mathbb{P}(\tilde{x}_{t+1} | x_t) \mathbb{P}(\tilde{y}_t | \bar{x}_t, \tilde{s}_t) \mathbb{P}(\tilde{s}_t | s_{t-1}) \mathbb{1}_{\{\tilde{i}_t^1 = i_t^1\}} \\ &= \mathbb{P}(\tilde{i}_{t+1}^1 | \tilde{i}_t^1, u_t) \end{aligned} \quad (19)$$

where we have added $\bar{x}_{0:t}$ in the conditioning in (a) because $\bar{x}_{0:t}$ is a deterministic function of $(x_{0:t}, u_{0:t})$ and (b) follows from the source and the channel models. By marginalizing (19), we get that for any $\tilde{i}_t^2 = (\tilde{s}_t, \tilde{y}_t, \tilde{i}_t^1)$, we have

$$\mathbb{P}(\tilde{i}_t^2 | i_t^1, u_t) = \mathbb{P}(\tilde{i}_t^2 | \tilde{i}_t^1, u_t) \quad (20)$$

Now, let $c(X_t, U_t, \hat{X}_t) = \lambda U_t + d(X_t, \hat{X}_t)$ denote the per-step cost. Recall that $\hat{X}_t = g_t(I_t^2)$. Thus, by (20), we get that

$$\mathbb{E}[c(X_t, U_t, \hat{X}_t) | i_t^1, u_t] = \mathbb{E}[c(X_t, U_t, \hat{X}_t) | \tilde{i}_t^1, u_t]. \quad (21)$$

Eq. (19) shows that $\{\tilde{I}_t^1\}_{t \geq 0}$ is a controlled Markov process controlled by $\{U_t\}_{t \geq 0}$. Eq. (21) shows that \tilde{I}_t^1 is sufficient for performance evaluation. Hence, by Markov decision theory [21], there is no loss of optimality in restricting attention to transmission strategies of the form (6).

B. Proof of Lemma 2

Consider

$$\begin{aligned} \pi_{t+1}^1(x_{t+1}) &= \mathbb{P}(x_{t+1} | s_{0:t}, y_{0:t}) \\ &= \sum_{x_t \in \mathcal{X}} \mathbb{P}(x_{t+1} | x_t) \mathbb{P}(x_t | s_{0:t}, y_{0:t}) \\ &= \sum_{x_t \in \mathcal{X}} P_{x_t x_{t+1}} \pi_t^2(x_t) = \pi_t^2 P \end{aligned} \quad (22)$$

which is the expression for $F^1(\cdot)$.

For F^2 , we consider the three cases separately. For $y_t \in \mathcal{X}$, we have

$$\pi_t^2(x) = \mathbb{P}(X_t = x | s_{0:t}, y_{0:t}) = \mathbb{1}_{\{x=y_t\}}. \quad (23)$$

For $y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\}$, we have

$$\begin{aligned} \pi_t^2(x) &= \mathbb{P}(X_t = x | s_{0:t}, y_{0:t}) \\ &= \frac{\mathbb{P}(X_t = x, y_t, s_t | s_{0:t-1}, y_{0:t-1})}{\mathbb{P}(y_t, s_t | s_{0:t-1}, y_{0:t-1})} \end{aligned} \quad (24)$$

Now, when $y_t = \mathfrak{E}_0$, we have that

$$\begin{aligned} \mathbb{P}(x_t, y_t, s_t | s_{0:t-1}, y_{0:t-1}) &= \mathbb{P}(y_t | x_t, \varphi_t(x_t), s_t) Q_{s_{t-1}s_t} \pi_t^1(x_t) \\ &\stackrel{(a)}{=} \begin{cases} Q_{s_{t-1}1} \pi_t^1(x_t), & \text{if } \varphi_t(x_t) = 0 \text{ and } s_t = 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned} \quad (25)$$

where (a) is obtained from the channel model. Substituting (25) in (24) and canceling $Q_{s_{t-1}1} \mathbb{1}_{\{s_t=1\}}$ from the numerator and the denominator, we get (recall that this is for the case when $y_t = \mathfrak{E}_0$),

$$\pi_t^2(x) = \frac{\mathbb{1}_{\{\varphi_t(x)=0\}} \pi_t^1(x)}{\pi_t^1(B_0(\varphi))}. \quad (26)$$

Similarly, when $y_t = \mathfrak{E}_1$, we have that

$$\mathbb{P}(x_t, y_t, s_t | s_{0:t-1}, y_{0:t-1}) = \mathbb{P}(y_t | x_t, \varphi_t(x_t), s_t) Q_{s_{t-1}s_t} \pi_t^1(x_t) \quad (27)$$

$$\stackrel{(b)}{=} \begin{cases} Q_{s_{t-1}0} \pi_t^1(x_t), & \text{if } s_t = 0 \\ 0, & \text{otherwise} \end{cases}$$

where (b) is obtained from the channel model. Substituting (27) in (24) and canceling $Q_{s_{t-1}0} \mathbb{1}_{\{s_t=0\}}$ from the numerator and the denominator, we get (recall that this is for the case when $y_t = \mathfrak{E}_1$),

$$\pi_t^2(x) = \pi_t^1(x). \quad (28)$$

By combining (23), (26) and (28), we get (9).

C. Proof of Theorem 1

Once we restrict attention to transmission strategies of the form (6), the information structure is partial history sharing [19]. Thus, one can use the common information approach of [19] and obtain the structure of optimal strategies.

Following [19], we split the information available at each agent into a “common information” and “local information”. Common information is the information available to all decision makers in the future; the remaining data at the decision maker is the local information. Thus, at the transmitter, the common information is $C_t^1 := \{S_{0:t-1}, Y_{0:t-1}\}$ and the local information is $L_t^1 := X_t$. Similarly, at the receiver, the common information is $C_t^2 := \{S_{0:t}, Y_{0:t}\}$ and the local information is $L_t^2 := \emptyset$. When the transmitter makes a decision, the state (sufficient for input output mapping) of the system is (X_t, S_{t-1}) ; when the receiver makes a decision, the state of the system is (X_t, S_t) . By [19, Proposition 1], we get that the sufficient statistic Θ_t^1 for the common information at the transmitter is

$$\Theta_t^1(x, s) = \mathbb{P}(X_t = x, S_{t-1} = s | S_{0:t-1}, Y_{0:t-1}),$$

and the sufficient statistic Θ_t^2 for the common information at the receiver is

$$\Theta_t^2(x, s) = \mathbb{P}(X_t = x, S_t = s | S_{0:t}, Y_{0:t}).$$

Note that Θ_t^1 is equivalent to (Π_t^1, S_{t-1}) and Θ_t^2 is equivalent to (Π_t^2, S_t) . Therefore, by [19, Theorem 2], there is no loss of optimality in restricting attention to transmission strategies of the form (10) and estimation strategies of the form

$$\hat{X}_t = g_t(S_t, \Pi_t^2). \quad (29)$$

Furthermore, the dynamic program of 1 follows from [19, Theorem 3].

Note that the right hand side of (14) implies that \hat{X}_t does not depend on S_t . Thus, instead of (29), we can restrict attention to estimation strategy of the form (11). Furthermore, the optimal estimation strategy is given by (15).

IV. PROOF OF OPTIMALITY OF THRESHOLD-BASED STRATEGIES FOR AUTOREGRESSIVE SOURCE

A. A change of variables

Define a process $\{Z_t\}_{t \geq 0}$ as follows: $Z_0 = 0$ and for $t \geq 0$,

$$Z_t = \begin{cases} aZ_{t-1}, & \text{if } Y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\} \\ Y_t, & \text{if } Y_t \in \mathcal{X} \end{cases}$$

Note that Z_t is a function of $Y_{0:t-1}$. Next, define processes $\{E_t\}_{t \geq 0}$, $\{E_t^+\}_{t \geq 0}$, and $\{\hat{E}_t\}_{t \geq 0}$ as follows:

$$E_t := X_t - aZ_{t-1}, \quad E_t^+ := X_t - Z_t, \quad \hat{E}_t := \hat{X}_t - Z_t$$

The processes $\{E_t\}_{t \geq 0}$ and $\{E_t^+\}_{t \geq 0}$ are related as follows: $E_0 = 0$, $E_0^+ = 0$, and for $t \geq 0$

$$E_t^+ = \begin{cases} E_t, & \text{if } Y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\} \\ 0, & \text{if } Y_t \in \mathcal{X} \end{cases}$$

and

$$E_{t+1} = aE_t^+ + W_t.$$

Since $X_t - \hat{X}_t = E_t^+ - \hat{E}_t$, we have that $d(X_t - \hat{X}_t) = d(E_t^+ - \hat{E}_t)$.

It turns out that it is easier to work with the processes $\{E_t\}_{t \geq 0}$, $\{E_t^+\}_{t \geq 0}$, and $\{\hat{E}_t\}_{t \geq 0}$ rather than $\{X_t\}_{t \geq 0}$ and $\{\hat{X}_t\}_{t \geq 0}$.

Next, redefine the pre- and post-transmission beliefs in terms of the error process. With a slight abuse of notation, we still denote the (probability density) of the pre- and post-transmission beliefs as π_t^1 and π_t^2 . In particular, π_t^1 is the conditional pdf of E_t given $(s_{0:t-1}, y_{0:t-1})$ and π_t^2 is the conditional pdf of E_t^+ given $(s_{0:t}, y_{0:t})$.

Let $H_t \in \{\mathfrak{E}_0, \mathfrak{E}_1, 1\}$ denote the event whether the transmission was successful or not. In particular,

$$H_t = \begin{cases} \mathfrak{E}_0, & \text{if } Y_t = \mathfrak{E}_0 \\ \mathfrak{E}_1, & \text{if } Y_t = \mathfrak{E}_1 \\ 1, & \text{if } Y_t \in \mathbb{R}. \end{cases}$$

We use h_t to denote the realization of H_t . Note that H_t is a deterministic function of U_t and S_t .

The time-evolutions of π_t^1 and π_t^2 is similar to Lemma 2. In particular, we have

Lemma 3 *Given any transmission strategy \mathbf{f} of the form (4):*

1) *there exists a function F^1 such that*

$$\pi_{t+1}^1 = F^1(\pi_t^2). \quad (30)$$

In particular,

$$\pi_{t+1}^1 = \begin{cases} \tilde{\pi}_t^2 \star \mu, & \text{if } y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\} \\ \mu, & \text{if } y_t \in \mathbb{R}, \end{cases} \quad (31)$$

where $\tilde{\pi}_t^2$ given by $\tilde{\pi}_t^2(e) := (1/|a|)\pi_t^2(e/a)$ is the conditional probability density of aE_t^+ , μ is the probability density function of W_t and \star is the convolution operation.

2) there exists a function F^2 such that

$$\pi_t^2 = F^2(\pi_t^1, \varphi_t, h_t). \quad (32)$$

In particular,

$$\pi_t^2 = \begin{cases} \delta_0, & \text{if } h_t = 1 \\ \pi_t^1|_{\varphi_t}, & \text{if } h_t = \mathfrak{E}_1 \\ \pi_t^1, & \text{if } h_t = \mathfrak{E}_0. \end{cases} \quad (33)$$

The key difference between Lemmas 2 and 3 (and the reason that we work with the error process $\{E_t\}_{t \geq 0}$ rather than $\{X_t\}_{t \geq 0}$) is that the function F^2 in (32) depends on h_t rather than y_t . Consequently, the dynamic program of Theorem 1 is now given by

$$V_{T+1}^1(s, \pi^1) = 0, \quad (34)$$

and for $t \in \{T, \dots, 0\}$

$$V_t^1(s, \pi^1) = \min_{\varphi: \mathbb{R} \rightarrow \{0,1\}} \left\{ \lambda \pi^1(B_1(\varphi)) + W_t^0(\pi^1, \varphi) \pi^1(B_0(\varphi)) + W_t^1(\pi^1, \varphi) \pi^1(B_1(\varphi)) \right\} \quad (35)$$

$$V_t^2(s, \pi^2) = D(\pi^2) + V_{t+1}^1(s, F^1(\pi^2)), \quad (36)$$

where,

$$W_t^0(\pi^1, \varphi) = Q_{s0} V_t^2(0, \pi^1) + Q_{s1} V_t^2(1, \pi^1|_{\varphi}),$$

$$W_t^1(\pi^1, \varphi) = Q_{s0} V_t^2(0, \pi^1) + Q_{s1} V_t^2(1, \delta_0),$$

$$D(\pi^2) = \min_{\hat{e} \in \mathbb{R}} \int_{\mathbb{R}} d(e - \hat{e}) \pi^2(e) de.$$

Again, note that due to the change of variables, the expression for W_t^1 does not depend on the transmitted symbol. Consequently, the expression for V_t^1 is simpler than that in Theorem 1.

B. Symmetric unimodal distributions and their properties

A probability density function π on reals is said to be *symmetric and unimodal* (SU) around $c \in \mathbb{R}$ if for any $x \in \mathbb{R}$, $\pi(c - x) = \pi(c + x)$ and π is non-decreasing in the interval $(-\infty, c]$ and non-increasing in the interval $[c, \infty)$.

Given $c \in \mathbb{R}$, a prescription $\varphi: \mathbb{R} \rightarrow \{0,1\}$ is called *threshold based around c* if there exists $k \in \mathbb{R}$ such that

$$\varphi(e) = \begin{cases} 1, & \text{if } |e - c| \geq k \\ 0, & \text{if } |e - c| < k. \end{cases}$$

Let $\mathcal{F}(c)$ denote the family of all threshold-based prescription around c .

Now, we state some properties of symmetric and unimodal distributions..

Property 1 If π is SU(c), then

$$c \in \arg \min_{\hat{e} \in \mathbb{R}} \int_{\mathbb{R}} d(e - \hat{e}) \pi(e) de.$$

For $c = 0$, the above property is a special case of [5, Lemma 12]. The result for general c follows from a change of variables.

Property 2 If π^1 is SU(0) and $\varphi \in \mathcal{F}(0)$, then for any $h \in \{\mathfrak{E}_0, \mathfrak{E}_1, 1\}$, $F^2(\pi^1, \varphi, h)$ is SU(0).

Proof: We prove the result for each $h \in \{\mathfrak{E}_0, \mathfrak{E}_1, 1\}$ separately. Recall the update of π^1 given by (33). For $h_t = \mathfrak{E}_0$, $\pi^2 = \pi^1$ and hence π^2 is SU(0). For $h_t = \mathfrak{E}_1$, $\pi^2 = \pi^1|_{\varphi}$; if $\varphi \in \mathcal{F}(0)$, then $\pi^1(x) \mathbb{1}_{\{\varphi(x)=0\}}$ is SU(0) and hence π^1 is SU(0). For $h_t = 1$, $\pi^2 = \delta_0$, which is SU(0). ■

Property 3 If π^2 is SU(0), then $F^1(\pi^2)$ is also SU(0).

Proof: Recall that F^1 is given by (31). The property follows from the fact that convolution of symmetric and unimodal distributions is symmetric and unimodal. ■

C. SU majorization and its properties

For any set \mathcal{A} , let $\mathcal{I}_{\mathcal{A}}$ denote its indicator function, i.e., $\mathcal{I}_{\mathcal{A}}(x)$ is 1 if $x \in \mathcal{A}$, else 0.

Let \mathcal{A} be a measurable set of finite Lebesgue measure, its *symmetric rearrangement* \mathcal{A}^σ is the open interval centered around origin whose Lebesgue measure is same as \mathcal{A} .

Given a function $\ell: \mathbb{R} \rightarrow \mathbb{R}$, its super-level set at level ρ , $\rho \in \mathbb{R}$, is $\{x \in \mathbb{R} : \ell(x) > \rho\}$. The *symmetric decreasing rearrangement* ℓ^σ of ℓ is a symmetric and decreasing function whose level sets are the same as ℓ , i.e.,

$$\ell^\sigma(x) = \int_0^\infty \mathcal{I}_{\{z \in \mathbb{R} : \ell(z) > \rho\}^\sigma}(x) d\rho.$$

Given two probability density functions ξ and π over \mathbb{R} , ξ *majorizes* π , which is denoted by $\xi \succeq_m \pi$, if for all $\rho \geq 0$,

$$\int_{|x| \geq \rho} \xi^\sigma(x) dx \geq \int_{|x| \geq \rho} \pi^\sigma(x) dx.$$

Given two probability density functions ξ and π over \mathbb{R} , ξ *SU majorizes* π , which we denote by $\xi \succeq_a \pi$, if ξ is SU and ξ majorizes π .

Now, we state some properties of SU majorization from [5].

Property 4 For any $\xi \succeq_a \pi$, where ξ is SU(c) and for any prescription φ , let $\theta \in \mathcal{F}(c)$ be a threshold-based prescription such that

$$\xi(B_i(\theta)) = \pi(B_i(\varphi)), \quad i \in \{0, 1\}.$$

Then, $\xi|_{\theta} \succeq_a \pi|_{\varphi}$. Consequently, for any $h \in \{\mathfrak{E}_0, \mathfrak{E}_1, 1\}$,

$$F^2(\xi, \theta, h) \succeq_a F^2(\pi, \varphi, h).$$

For $c = 0$, the result follows from [5, Lemma 7 and 8]. The result for general c follows from change of variables.

Property 5 For any $\xi \succeq_m \pi$, $F^1(\xi) \succeq_a F^1(\pi)$.

This follows from [5, Lemma 10].

Recall the definition of $D(\pi^2)$ given after (36).

Property 6 If $\xi \succeq_a \pi$, then

$$D(\pi) \geq D(\pi^\sigma) \geq D(\xi^\sigma) = D(\xi).$$

This follows from [5, Lemma 11].

D. Qualitative properties of the value function and optimal strategy

Lemma 4 *The value functions V_t^1 and V_t^2 of (34)–(36), satisfy the following property.*

(P1) *For any $i \in \{1, 2\}$, $s \in \{0, 1\}$, $t \in \{0, \dots, T\}$, and pdfs ξ^i and π^i such that $\xi^i \succeq_a \pi^i$, we have that $V_t^i(s, \xi^i) \leq V_t^i(s, \pi^i)$.*

Furthermore, the optimal strategy satisfies the following properties. For any $s \in \{0, 1\}$ and $t \in \{0, \dots, T\}$:

(P2) *if π^1 is SU(c), then there exists a prescription $\varphi_t \in \mathcal{F}(c)$ that is optimal. In general, φ_t depends on π^1 .*

(P3) *if π^2 is SU(c), then the optimal estimate \hat{E}_t is c .*

Proof: We proceed by backward induction. $V_{T+1}^1(s, \pi^1)$ trivially satisfies the (P1). This forms the basis of induction. Now assume that $V_{t+1}^1(s, \pi^1)$ also satisfies (P1). For $\xi^2 \succeq_a \pi^2$, we have that

$$\begin{aligned} V_t^2(s, \pi^2) &= D(\pi^2) + V_{t+1}^1(s, F^1(\pi^2)) \\ &\stackrel{(a)}{\geq} D(\xi^2) + V_{t+1}^1(s, F^1(\xi^2)) \\ &= V_t^2(s, \xi^2), \end{aligned} \quad (37)$$

where (a) follows from Properties 5 and 6 and the induction hypothesis. Eq. 37 implies that V_t^2 also satisfies (P1).

Now, consider $\xi^1 \succeq_a \pi^1$. Let φ be the optimal prescription at π^1 . Let θ be the threshold-based prescription corresponding to φ as defined in Property 3. By construction,

$$\pi^1(B_0(\varphi)) = \xi^1(B_0(\theta)) \quad \text{and} \quad \pi^1(B_1(\varphi)) = \xi^1(B_1(\theta)).$$

Moreover, from Property 3 and (37),

$$W_t^0(\pi^1, \varphi) \geq W_t^0(\xi^1, \theta) \quad \text{and} \quad W_t^1(\pi^1, \varphi) \geq W_t^1(\xi^1, \theta).$$

Combining the above two equations with (35), we get

$$\begin{aligned} V_t^1(s, \pi^1) &= \lambda \pi^1(B_1(\varphi)) + W_t^0(\pi^1, \varphi) \pi^1(B_0(\varphi)) \\ &\quad + W_t^1(\pi^1, \varphi) \pi^1(B_1(\varphi)) \\ &\geq \lambda \xi^1(B_1(\theta)) + W_t^0(\xi^1, \theta) \xi^1(B_0(\theta)) \\ &\quad + W_t^1(\xi^1, \theta) \xi^1(B_1(\theta)) \\ &\geq V_t^1(s, \xi^1) \end{aligned} \quad (38)$$

where the last inequality follows by minimizing over all θ . Eq. (38) implies that V_t^1 also satisfies (P1). Hence, by the principle of induction, (P1) is satisfied for all time.

The argument in (38) also implies (P2). Furthermore, (P3) follows from Property 1. ■

E. Proof of Theorem 2

We first prove a weaker version of the structure of optimal transmission strategies. In particular, there exist threshold functions $\tilde{k}_t: \{0, 1\} \times \Delta(\mathbb{R}) \rightarrow \mathbb{R}_{\geq 0}$ such that the following transmission strategy is optimal:

$$f_t(X_t, S_{t-1}, \Pi_t^1) = \begin{cases} 1, & \text{if } |X_t - aZ_{t-1}| \geq \tilde{k}_t(S_{t-1}, \Pi_t^1) \\ 0, & \text{otherwise.} \end{cases} \quad (39)$$

or, equivalently, in terms of the $\{E_t\}_{t \geq 0}$ process:

$$f_t(E_t, S_{t-1}, \Pi_t^1) = \begin{cases} 1, & \text{if } |E_t| \geq \tilde{k}_t(S_{t-1}, \Pi_t^1) \\ 0, & \text{otherwise.} \end{cases} \quad (40)$$

We prove (40) by induction. Note that $\pi_0^1 = \delta_0$ which is SU(0). Therefore, by (P2), there exists a threshold-based prescription $\varphi_0 \in \mathcal{F}(0)$ that is optimal. This forms the basis of induction. Now assume that until time $t-1$, all prescriptions are in $\mathcal{F}(0)$. By Properties 2 and 3, Π_t^1 is SU(0). Therefore, by (P2), there exists a threshold-based prescription $\varphi_t \in \mathcal{F}(0)$ that is optimal. This proves the induction step and, hence, by the principle of induction, threshold-based prescriptions of the form (40) are optimal for all time. Translating the result back to $\{X_t\}_{t \geq 0}$, we get that threshold-based prescriptions of the form (39) are optimal.

Observe that Properties 2 and 3 also imply that for all t , Π_t^2 is SU(0). Therefore, by Property 1, the optimal estimate $\hat{E}_t = 0$. Recall that $\hat{E}_t = \hat{X}_t - Z_t$. Thus, $\hat{X}_t = Z_t$. This proves the first part of Theorem 2.

To prove that there exist optimal transmission strategies where the thresholds do not depend on Π_t^1 , we fix the estimation strategy to be of the form (17) and consider the problem of finding the best transmission strategy at the sensor. This is a single-agent (centralized) stochastic control problem and the optimal solution is given by the following dynamic program:

$$J_{T+1}(e, s) = 0 \quad (41)$$

and for $t \in \{T, \dots, 0\}$

$$J_t(e, s) = \min\{J_t^0(e, s), J_t^1(e, s)\} \quad (42)$$

where

$$\begin{aligned} J_t^0(e, s) &= d(e) + Q_{s0} \mathbb{E}_W[J_{t+1}(ae + W, 0)] \\ &\quad + Q_{s1} \mathbb{E}_W[J_{t+1}(ae + W, 1)], \end{aligned} \quad (43)$$

$$\begin{aligned} J_t^1(e, s) &= \lambda + Q_{s0} d(e) + Q_{s0} \mathbb{E}_W[J_{t+1}(ae + W, 0)] \\ &\quad + Q_{s1} \mathbb{E}_W[J_{t+1}(W, 1)], \end{aligned} \quad (44)$$

We now use the results of [22] to show that the value function even and increasing on $\mathbb{R}_{\geq 0}$ (abbreviated to EI).

The results of [22] rely on stochastic dominance. Given two probability density functions ξ and π over $\mathbb{R}_{\geq 0}$, ξ *stochastically dominates* π , which we denote by $\xi \succeq_s \pi$, if

$$\int_{x \geq y} \xi(x) dx \geq \int_{x \geq y} \pi(x) dx, \quad \forall y \in \mathbb{R}_{\geq 0}.$$

Now, we show that dynamic program (41)–(44) satisfies conditions (C1)–(C3) of [22, Theorem 1]. In particular, we have: Condition (C1) is satisfied because the per-step cost functions $d(e)$ and $\lambda + Q_{s0}d(e)$ are EI. Condition (C2) is satisfied because the probability density μ of W_t is even, which implies that for any $e \in \mathbb{R}_{\geq 0}$,

$$\int_{w \in \mathbb{R}} \mu(ae + w) dw = \int_{w \in \mathbb{R}} \mu(-ae + w) dw.$$

Now, to check condition (C3), define for $e \in \mathbb{R}$ and $y \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} M^0(y|e) &= \int_y^\infty \mu(ae + w)dw + \int_{-\infty}^{-y} \mu(ae + w)dw \\ &= 1 - \int_{-y}^y \mu(ae + w)dw, \\ M^1(y|e) &= \int_y^\infty \mu(w)dw + \int_{-\infty}^{-y} \mu(w)dw. \end{aligned}$$

$M^1(y|e)$ does not depend on e and is thus trivially even and increasing in e . Since μ is even, $M^0(y|e)$ is even in e . We show that $M^0(y|e)$ is increasing in e for $e \in \mathbb{R}_{\geq 0}$ later (see Lemma 5).

Since conditions (C1)–(C3) of [22, Theorem 1] are satisfied, we have that for any $s \in \{0, 1\}$, $J_t(e, s)$ is even in e and increasing for $e \in \mathbb{R}_{\geq 0}$. Now, observe that

$$\begin{aligned} J^0(e, s) - J^1(e, s) &= (1 - Q_{s0})d(e) + Q_{s1}\mathbb{E}_W[J_{t+1}(ae + W, 1)] \\ &\quad - \lambda - Q_{s1}\mathbb{E}_W[J_{t+1}(W, 1)] \end{aligned}$$

which is even in e and increasing in $e \in \mathbb{R}_{\geq 0}$. Therefore, for any fixed $s \in \{0, 1\}$, the set A of e in which $J_t^0(e, s) - J_t^1(e, s) \leq 0$ is convex and symmetric around the origin, i.e., a set of the form $[-k_t(s), k_t(s)]$. Thus, there exist a $k_t(\cdot)$ such that the action $u_t = 0$ is optimal for $e \in [-k_t(s), k_t(s)]$. This, proves the structure of the optimal transmission strategy.

Lemma 5 *For any $y \in \mathbb{R}_{\geq 0}$, $M^0(y|e)$ is increasing in e , $e \in \mathbb{R}_{\geq 0}$.*

Proof: To show that $M^0(y|e)$ is increasing in e for $e \in \mathbb{R}_{\geq 0}$, it suffices to show that $1 - M^0(y|e) = \int_{-y}^y \mu(ae + w)dw$ is decreasing in e for $e \in \mathbb{R}_{\geq 0}$. Consider a change of variables $x = ae + w$. Then,

$$1 - M^0(y|e) = \int_{-y}^y \mu(ae + w)dw = \int_{-y-ae}^{y-ae} \mu(x)dx \quad (45)$$

Taking derivative with respect to e , we get that

$$\frac{\partial M^0(y|e)}{\partial e} = a[\mu(y - ae) - \mu(-y - ae)] \quad (46)$$

Now consider the following cases:

- If $a > 0$ and $y > ae > 0$, then the right hand side of (46) equals $a[\mu(y - ae) - \mu(y + ae)]$, which is positive.
- If $a > 0$ and $ae > y > 0$, then the right hand side of (46) equals $a[\mu(ae - y) - \mu(ae + y)]$, which is positive.
- If $a < 0$ and $y > |a|e > 0$, then the right hand side of (46) equals $|a|[\mu(y - |a|e) - \mu(y + |a|e)]$, which is positive.
- If $a < 0$ and $|a|e > y > 0$, then the right hand side of (46) equals $|a|[\mu(|a|e - y) - \mu(|a|e + y)]$, which is positive.

Thus, in all cases, $M^0(y|e)$ is increasing in e , $e \in \mathbb{R}_{\geq 0}$. ■

V. CONCLUSION

In this paper, we studied remote estimation over a Gilbert-Elliott channel with feedback. We assume that the channel state is observed by the receiver and fed back to the transmitter with one unit delay. In addition, the transmitter gets ACK/NACK feedback for successful/unsuccessful transmission. Using ideas from team theory, we establish the structure of optimal transmission and estimation strategies and identify a dynamic program to determine optimal strategies with that structure. We then consider first-order autoregressive sources where the noise process has unimodal and symmetric distribution. Using ideas from majorization theory, we show that the optimal transmission strategy has a threshold structure and the optimal estimation strategy is Kalman-like.

A natural question is how to determine the optimal thresholds. For finite horizon setup, these can be determined using the dynamic program of (41)–(44). For infinite horizon setup, we expect that the optimal threshold will not depend on time. We believe that it should be possible to evaluate the performance of a generic threshold based strategy using an argument similar to the renewal theory based argument presented in [16] for channels without packet drops.

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